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Relative Asymptotics for Orthogonal Matrix Polynomials with Convergent Recurrence Coefficients

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The asymptotic behavior of $\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}$ and $P_n(x, d\beta)\,P_n^{-1}(x, d\alpha)$ is studied. Here $(\gamma_n(.))_n$ are the leading coefficients of the orthonormal matrix polynomials $P_n(x, .)$ with respect to the matrix measures $d\beta$ and $d\alpha$ which are related by $d\beta(u) = d\alpha(u) + \sum_{k=1}^{N} M_k \delta(u - c_k)$, where M_k are positive definite matrices, δ is the Dirac measure and c_k lies outside the support of $d\alpha$ for $k = 1, ..., N$. Finally, we deduce the asymptotic behavior of $P_n(c, d\beta)\,MP_n^*(c, d\alpha)$ when $d\beta(u) = d\alpha(u) + M\delta(u - c)$, with M a positive definite matrix and c outside the support of $d\alpha$.

Key Words: ratio asymptotics; orthogonal matrix polynomials; perturbation of matrix measure.

1. INTRODUCTION

Let $\mathscr{B}$ be a σ-algebra of subsets of the space Ω, and let $\rho = [\rho_{i,j}]$ be a $N \times N$ non-negative matrix valued function on $\mathscr{B}$ which is countably additive, i.e., each entry function $\rho_{i,j}$ is countably additive on $\mathscr{B}$. We shall refer to ρ as a non-negative valued measure on $(\Omega, \mathscr{B})$.

Let Φ be a matrix-valued function on Ω and ν be a non-negative real valued measure on $\mathscr{B}$. If Φ is $\mathscr{B}$-measurable, i.e., if each of its entries $\varphi_{i,j}$

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is $\mathscr{B}$-measurable, then we say that Φ is integrable and we define $\int_\Omega \Phi\, dv = [\int_\Omega \varphi_{i,j}\, dv]$.

From the fact that ρ is a non-negative valued matrix, whose entries are non-negative real valued measures, we have $0 \leqslant \rho \leqslant \tau\rho\, I$ ($A \geqslant B$ means that $A-B$ is a positive semi-definite matrix, and $\tau\rho$ is the trace measure of ρ). It follows that each $\rho_{i,j}$ is absolutely continuous with respect to the measure $\tau\rho$, and the trace derivative $\rho'_\tau = [d\rho_{i,j}/d\tau\rho]_{1\leqslant i,j\leqslant N}$ is $\mathscr{B}$-measurable, satisfying $0 \leqslant \rho'_\tau \leqslant I$ $\tau\rho$-almost everywhere. Consequently from the Radon-Nikodym theorem we have $\rho(A) = \int_A \rho'_\tau(w)\, d\tau\rho$, for each $A \in \mathscr{B}$. Let Φ and Ψ be two $N\times N$ real matrix-valued functions on Ω. We say that (Φ, Ψ) is integrable with respect to the non-negative valued measure ρ if $\Phi\rho'_\tau\Psi^*$ is integrable with respect to the real measure $\tau\rho$ and we define $\int_\omega \Phi\, d\rho \Psi^* = \int_\omega \Phi\rho'_\tau\Psi^*\, d\tau\rho$.

We shall consider in the linear space of the polynomials $\mathbb{C}^{N\times N}[t]$ in the variable t with matrix coefficients in $\mathbb{C}^{N\times N}$, an inner product defined in the following way

$$\langle P, Q\rangle_{d\alpha} \overset{\text{def}}{=} \int_\Omega P(t)\, d\alpha(t)\, Q^*(t), \tag{1}$$

with $\Omega = \mathbb{R}$, and α a positive definite valued matrix measure whose entries are Borel real measures (see [15]).

A generalization of the Gram-Schmidt orthonormalization procedure for the set $\{I, xI, x^2I, ...\}$ with respect to (1) will give a set of orthonormal matrix polynomials $(P_n(., d\alpha))_n$ which satisfies

$$\langle P_n(., d\alpha), P_m(., d\alpha)\rangle_{d\alpha} = \int_{\mathbb{R}} P_n(x, d\alpha)\, \alpha'(x)\, P_m^*(x, d\alpha)\, d\tau\alpha = \delta_{n,m} I.$$

Notice that the set $(U_n P_n(., d\alpha))_n$ is also a set of orthonormal matrix polynomials for every sequence of unitary matrices $(U_n)_n$.

Orthogonal matrix polynomials have been studied in the second half of this century. Krein obtained some results about matrix moment problems from the point of view of operator theory [11]. Recently, during the 80's they have been connected to scattering theory by Geronimo [7], and an analog of Favard's theorem has been established for three-term recurrence matrix relation by A. I. Aptekarev and E. M. Nikishin [1]. Some algebraic results and results concerning the zeros were found by D. Zhani [19]. More recently some results concerning zeros and quadrature formulae have been studied by A. Sinap and W. Van Assche [17] and finally some results concerning zeros, quadrature formulae, asymptotic behavior of orthogonal polynomials have been obtained by A. Durán and coworkers [3, 5, 6].

Matrix orthogonal polynomials appear in a natural way when we consider different kinds of non-standard inner products.

In [12] it is shown that a family of such orthogonal polynomials for matrix measures supported on the real line can be obtained from a sequence of polynomials orthogonal with respect to a scalar measure supported on an harmonic algebraic curve.

In [13] the same problem is studied when we consider polynomials orthogonal with respect to a scalar measure supported on a lemniscate. There, a class of a matrix orthogonal polynomials on the unit circle is obtained.

In [4], Sobolev-type polynomials are analyzed from the perspective of a decomposition of them. In such a way one can associate to them a matrix measure plus some matrix mass points. Again, matrix orthogonal polynomials on the real line appear. Thus, the knowledge of properties of matrix orthogonal polynomials, both analytic and algebraic, is a basic tool in order to obtain results for the non-standard scalar cases pointed out above.

In [9] Gonchar studies Padé approximation to Markov functions to which a rational function is added, which is the same as adding mass points to a given measure.

The aim of our work is to extend to the matrix case some asymptotic results obtained by P. Nevai [14, Lemma 16, p. 132] when some mass points are added to a measure such that the corresponding Jacobi matrix is a compact perturbation of the infinite tridiagonal matrix

$$\begin{pmatrix} b & a & & & \\ a & b & a & & \\ & a & b & a & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

This kind of asymptotic results has been obtained, separately, in [9] in the framework of rational approximation. In fact, the approximation to some classes of meromorphic functions ("Markov functions" with a perturbation by a rational function with prescribed poles) is considered, and the relative asymptotics for the denominators of the corresponding Padé approximants is obtained (see [9]).

In this paper we shall consider two matrix measures $d\alpha$ and $d\beta$ such that $d\beta(u) = d\alpha(u) + \sum_{k=1}^{N} M_k \delta(u - c_k)$, where M_k is a positive definite matrix, δ is the Dirac matrix measure and c_k lies outside the support of $d\alpha$ for $k = 1, ..., N$.

Let $(P_n(x, .) = \gamma_n(.)\, x^n + b_n(.)\, x^{n-1} + \textit{lower degree terms})_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$ and $d\beta$, respectively.

In Section 2 we will introduce orthogonal matrix polynomials on the real line and discuss some properties which we will need in the next sections. In Section 3 we deduce the behavior of $\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}$. In Section 4 we give the relative asymptotics for some nth orthonormal matrix polynomials with respect to the matrix measure $d\beta$ and the nth orthonormal matrix polynomials with respect to the matrix measure $d\alpha$.

2. ORTHOGONAL MATRIX POLYNOMIALS ON THE REAL LINE

For each real matrix measure $d\alpha$, we introduce

$$\langle P, Q\rangle_{d\alpha} \stackrel{\text{def}}{=} \int_{\mathbb{R}} P(x)\, d\alpha Q^*(x). \tag{2}$$

This bilinear form satisfies

1. $\langle P, Q\rangle_{d\alpha} = \langle Q, P\rangle^*_{d\alpha}$.
2. $\langle xP, Q\rangle_{d\alpha} = \langle P, xQ\rangle_{d\alpha}$.
3. $\langle P, P\rangle_{d\alpha}$ is a non-negative definite matrix. If $\det P \neq 0$, it is a positive definite matrix.

In the following, we consider a real matrix measure $d\alpha$ for which $\langle P, P\rangle_{d\alpha} \neq 0$ for every matrix polynomial P with non-singular leading coefficient.

As in the scalar case, the orthonormal matrix polynomials $(P_n(x, d\alpha))_n$ with respect to the matrix measure $d\alpha$ are orthogonal to every matrix polynomial of degree less than n and they satisfy a three-term recurrence relation

$$\begin{aligned} xP_n(x, d\alpha) &= D_{n+1}(d\alpha)\, P_{n+1}(x, d\alpha) \\ &\quad + E_n(d\alpha)\, P_n(x, d\alpha) + D_n^*(d\alpha)\, P_{n-1}(x, d\alpha), \end{aligned} \tag{3}$$

where $P_{-1} = 0$, $P_0 = \int d\alpha = I$, $D_n(d\alpha) = \gamma_{n-1}(d\alpha)\, \gamma_n^{-1}(d\alpha)$ is a positive definite matrix ($\gamma_n(d\alpha)$ is the leading coefficient of $P_n(x, d\alpha)$) and $E_n(d\alpha)$ is an hermitian matrix. The corresponding matrix polynomials of the second kind are defined by

$$Q_n(x, d\alpha) \stackrel{\text{def}}{=} \int \frac{P_n(x, d\alpha) - P_n(t, d\alpha)}{x - t}\, d\alpha(t). \tag{4}$$

These matrix polynomials satisfy the following Liouville–Ostrogradski formula

$$Q_n(x, d\alpha)\, P^*_{n-1}(x, d\alpha) - P_n(x, d\alpha)\, Q^*_{n-1}(x, d\alpha) = D_n^{-1}(d\alpha). \tag{5}$$

Furthermore, defining

$$K_{n+1}(x, y, d\alpha) \stackrel{\text{def}}{=} \sum_{j=0}^{n} P_j^*(y, d\alpha)\, P_j(x, d\alpha),$$

we get the Christoffel-Darboux formula

$$\begin{aligned}(x-y)\, K_{n+1}(x, y, d\alpha) = P_n^*(y, d\alpha)\, D_{n+1}(d\alpha)\, P_{n+1}(x, d\alpha)\\ - P_{n+1}^*(y, d\alpha)\, D_{n+1}^*(d\alpha)\, P_n(x, d\alpha).\end{aligned} \tag{6}$$

By means of a straightforward computation we get the following equation

$$\begin{aligned}K_{n+1}(x, x, d\alpha) = P_{n+1}^*(x, d\alpha)'\, D_{n+1}^*(d\alpha)\, P_n(x, d\alpha)\\ - P_n^*(x, d\alpha)'\, D_{n+1}(d\alpha)\, P_{n+1}(x, d\alpha).\end{aligned} \tag{7}$$

The matrix $K_n(x, y, d\alpha)$ is called the nth reproducing kernel because of the following property. For every matrix polynomial $\Pi_m(x)$ of degree $m \leqslant n-1$, we have

$$\langle \Pi_m, K_n(\cdot, y, d\alpha)\rangle_{d\alpha} = \int_{\mathbb{R}} \Pi_m(x)\, d\alpha K_n^*(x, y, d\alpha) = \Pi_m(y). \tag{8}$$

In the next section, we will use the following results (see [6]) concerning ratio asymptotic properties for orthogonal matrix polynomials.

First we start with some definitions. Let Δ_n be the set of zeros of the matrix polynomial P_n, i.e.,

$$\Delta_n(d\alpha) \stackrel{\text{def}}{=} \{x_{n,k};\, k = 1, ..., N\colon \det P_n(x_{n,k}, d\alpha) = 0\},$$

and setting

$$\Gamma = \bigcap_{N>0} M_N \qquad \text{where} \qquad M_N = \overline{\bigcup_{n \geqslant N} \Delta_n(d\alpha)},$$

we have $supp(d\alpha) \subset \Gamma$. Here we define the support of $d\alpha$ as $supp(d\alpha) = supp(\tau d\alpha) = supp(d\alpha_{1,1} + d\alpha_{2,2} + \cdots + d\alpha_{N,N})$.

We recall that if H is a positive definite (resp. positive semidefinite) matrix then there is a unique square root $H_0 = H^{1/2}$ of H defined as follows. Writing $H = UDU^*$, where $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ and $(\lambda_i)_{i=1,n}$ are the positive eigenvalues of H then $H_0 = UD_0U^*$, where $D_0 = diag(+\sqrt{\lambda_1}, +\sqrt{\lambda_2}, ..., +\sqrt{\lambda_n})$.

Proposition 2.1 [6]. *Let $(P_n(x, d\alpha))_n$ be a sequence of orthonormal matrix polynomials satisfying the three-term recurrence relation* (3). *Assume* *that* $\lim_{n\to\infty} D_n(d\alpha)=D$ *and* $\lim_{n\to\infty} E_n(d\alpha)=E$, *with* D *a non-singular matrix then*

$$\lim_{n\to\infty} P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha)\, D_n^{-1}(d\alpha)=\int \frac{dW_{D,E}(t)}{z-t}; \quad z\in\mathbb{C}\setminus\Gamma, \tag{9}$$

where $W_{D,E}(t)$ *is the matrix weight for the Chebyshev matrix polynomials of the second kind. Moreover the convergence is uniform on compact subsets of* $\mathbb{C}\setminus\Gamma$.

The analytic function on the right-hand side of (9) *has the following explicit form*

- *If* D *is an hermitian positive definite matrix, then*

$$\int \frac{dW_{D,E}(t)}{x-t}=\frac{1}{2}\, D^{-1}(xI-E)\, D^{-1} -\frac{1}{2}\, D^{-1/2}\big[\sqrt{D^{-1/2}(E-xI)\, D^{-1}(E-xI)\, D^{-1/2}-4I}\,\big]\, D^{-1/2}, \tag{10}$$

where $x\notin supp(dW_{D,E})$.

Here, $supp(dW_{D,E})=\{x\in\mathbb{R};\ D^{-1/2}(E-xI)\, D^{-1/2}$ *has at least one eigenvalue in* $[-2, 2]\}$.

- *If* D *is an hermitian matrix, then*

$$\int \frac{dW_{D,E}(t)}{x-t}=\frac{1}{2}\, D^{-1}(xI-E)\, D^{-1}-\frac{1}{2}\, D^{-1}(E-xI)^{1/2} \times\big[\sqrt{I-4(E-xI)^{-1/2}\, D(E-xI)^{-1}\, D(E-xI)^{-1/2}}\,\big] \times(E-xI)^{1/2}\, D^{-1}, \tag{11}$$

where $x\notin supp(dW_{D,E})$.

Here, $supp(dW_{D,E})=\{x\in\mathbb{R}\setminus[b_1, b_N];\ (E-xI)^{1/2}\, D^{-1}(E-xI)^{1/2}$ *has at least one eigenvalue in* $[-2, 2]\}$, *and* $b_1\leqslant b_2\leqslant\ldots\leqslant b_N$ *are the eigenvalues of* E.

LEMMA 2.1. *Let* $(P_n(x, d\alpha))_n$ *be a sequence of orthonormal matrix polynomials with convergent recurrence coefficients which appear in* (3). *There exists a positive constant* $a>0$ *such that if* $x_{n,k}$ *is a zero of* P_n *then* $|x_{n,k}|\leqslant a$, *and* $supp(d\alpha)$ *is contained in* $[-a, a]$.

Proof. Let

$$J \stackrel{\text{def}}{=} \begin{pmatrix} E_0 & D_1 & 0 & \cdots & \cdots \\ D_1^* & E_1 & D_1 & 0 & \cdots \\ 0 & D_2^* & E_2 & D_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

be the Jacobi matrix with convergent matrix parameters $\{E_i, D_{i+1}\}_{i=0}^{\infty}$. It is known that the zeros $(x_{n,k})_{k=1}^{nN}$ of P_n are eigenvalues of J_{nN} (J_{nN} is the truncated Jacobi matrix of dimension nN) (see [5, 17]). Using the Gershgorin disks for the location of eigenvalues, there is a positive number a such that $|x_{n,k}| \leqslant a$, and therefore $supp(d\alpha) \subset \Gamma \subset [-a, a]$. ∎

In the following, we denote by $\hat{\Gamma}$ the smallest closed interval which contains the support of $d\alpha$.

PROPOSITION 2.2. *The Markov matrix function* $\int (dW_{D,E}(t))/(z-t)$ *is positive or negative definite for each* $z \in \mathbb{R} \backslash \hat{\Gamma}$. *It is differentiable and its derivative* $-\int (dW_{D,E}(t))/((z-t)^2)$ *is negative definite when* z *is outside* $supp(dW_{D,E})$.

Proof. We consider

$$A = \{x \in \mathbb{R} : x > c;\ \forall c \in \hat{\Gamma}\}$$

and

$$B = \{x \in \mathbb{R} : x < c;\ \forall c \in \hat{\Gamma}\}.$$

The function $\frac{1}{z-t}$ is positive (resp. negative) when $z \in A$ (resp. $z \in B$) and $t \in supp(dW_{D,E}) = supp(d\alpha) \subset \hat{\Gamma}$. Since $dW_{D,E}$ is a positive definite matrix measure, the trace derivative $W'_{D,E}(t) = [dW_{D,E}/d\tau W_{D,E}]$ is positive definite ($d\tau W_{D,E}$ is the positive real trace measure of $W_{D,E}$), and for any real vector $u \in \mathbb{C}^N$, we have

$$u\left(\int \frac{dW_{D,E}(t)}{z-t}\right) u^* = u\left(\int \frac{W'_{D,E}(t)}{z-t}\, d\tau W_{D,E}\right) u^*$$

$$= \int \frac{(uW'_{D,E}(t)\, u^*)}{z-t}\, d\tau W_{D,E}.$$

Consequently the Markov matrix function is positive (resp. negative) definite in $z \in A$ (resp. $z \in B$).

In a similar way, we can show that the derivative of the Markov matrix function $\frac{d}{dz}\int (dW_{D,E}(t))/(z-t) = -\int (dW_{D,E}(t))/((z-t)^2)$ is negative definite when $z \notin supp(dW_{D,E})$. ∎

In the next sections, we will need the asymptotics of the derivative of $(P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha))_n$ where z belongs to a compact subset of $\mathbb{C}\backslash\Gamma$. To do that we use the well known theorem of Weierstrass concerning the differentiation of a sequence of holomorphic functions. We recall that $\{W_n\}_n$ is locally uniformly convergent in the open set G to a function W ($W_n(z) \rightarrowtail W(z)$) if $W_n(z)$ is uniformly convergent to $W(z)$ in every closed set contained in G.

THEOREM 2.1. *If a sequence $\{W_n\}_n$ of functions holomorphic in an open set G is locally uniformly convergent in G to a function W, then the function W is also holomorphic in G, and if $\infty \notin G$, then*

$$W_n^{(k)}(z) \rightarrowtail W^{(k)}(z)$$

in the set G for $k = 1, 2, \ldots$.

Proof. See [16]. ∎

COROLLARY 2.1. *Under the hypothesis of Proposition* 2.1 *we have*

$$(P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha))^{(k)} D_n^{-1} \rightarrowtail \left(\int \frac{dW_{D,E}(t)}{z-t}\right)^{(k)} \tag{12}$$

on compact subsets of $\mathbb{C}\backslash\Gamma$, for $k = 1, 2, \ldots$.

The local uniform convergence in (12) means that every entry of the left hand-side of (12) is locally uniformly convergent to its corresponding entry in the right hand-side of (12).

Proof. Since $(P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha))$ is an holomorphic matrix function in $\mathbb{C}\backslash\Gamma$, each one of its entries is an holomorphic function in $\mathbb{C}\backslash\Gamma$, and from the hypothesis we have that $(P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha)\, D_n)_n$ converges uniformly on compact subsets of $\mathbb{C}\backslash\Gamma$, hence for each one of its entries. Therefore $(P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha))_n$ is locally uniformly convergent in compact subsets of $\mathbb{C}\backslash\Gamma$ since each closed subset of a compact set is compact.

Applying Theorem 2.1 we get

$$(P_{n-1}(z, d\alpha)\, P_n^{-1}(z, d\alpha))^{(k)} D_n^{-1} \rightarrowtail \left(\int \frac{dW_{D,E}(t)}{z-t}\right)^{(k)}. \quad ∎$$

3. THE RATIO ASYMPTOTICS FOR LEADING COEFFICIENTS

Let $d\alpha$ and $d\beta$ be two matrices of measures, such that $d\beta(u) = d\alpha(u) + M\delta(u-c)$, $c \in \mathbb{R}\backslash\hat{\Gamma}$, where M is a positive definite matrix. Let $(P_n(x, .) = \gamma_n(.)\, x^n + b_n(.)\, x^{n-1} + \textit{lower degree terms})_n$, be a sequence of orthonormal matrix polynomials with respect to the matrix measures $d\alpha$ and $d\beta$. We assume that

$$\lim_{n \to \infty} D_n(d\alpha) = D \qquad \text{and} \qquad \lim_{n \to \infty} E_n(d\alpha) = E, \tag{13}$$

where $D_n(d\alpha)$ and $E_n(d\alpha)$ are the matrix coefficients in the recurrence formula (3) and D is a non-singular matrix.

In this section, we study the asymptotic behavior of the ratio of the leading coefficient of some nth orthonormal matrix polynomials associated to $d\beta$ and the leading coefficient of the nth orthonormal matrix polynomials associated $d\alpha$. We start with the following lemma which contains some formulas relating the sequences of orthonormal polynomials $(P_n(x, d\alpha))_n$ and $(P_n(x, d\beta))_n$.

LEMMA 3.1. *Let $d\alpha$ and $d\beta$ be two matrix measures, and M be a positive definite matrix such that $d\beta(u) = d\alpha(u) + M\delta(u-c)$, where c is a real number, Then*

$$\begin{aligned} P_n(x, d\beta) = \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha) \times [&P_n(x, d\alpha) \\ &- P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MK_{n+1}^*(c, x, d\alpha)]. \end{aligned} \tag{14}$$

Proof. Using the reproducing property (8) for the kernel polynomials, we have

$$\begin{aligned} P_n(x, d\beta) &= \int P_n(u, d\beta)\, d\alpha(u)\, K_{n+1}^*(u, x, d\alpha) \\ &= \int P_n(u, d\beta)\, d\beta(u)\, K_{n+1}^*(u, x, d\alpha) \\ &\quad - \int P_n(u, d\beta)\, MK_{n+1}^*(u, x, d\alpha)\, d\delta(u-c) \\ &= \sum_{j=0}^{n} \left[\int P_n(u, d\beta)\, d\beta(u)\, P_j^*(u, d\alpha) \right] P_j(x, d\alpha) \\ &\quad - P_n(c, d\beta)\, MK_{n+1}^*(c, x, d\alpha) \end{aligned}$$

$$= \int P_n(u, d\beta)\, d\beta(u)\, P_n^*(u, d\alpha) . P_n(x, d\alpha)$$

$$- P_n(c, d\beta)\, MK_{n+1}^*(c, x, d\alpha)$$

$$= \gamma_n(d\beta)^{-*}\, \gamma_n(d\alpha)^*\, P_n(x, d\alpha) - P_n(c, d\beta)\, MK_{n+1}^*(c, x, d\alpha).$$

If $x = c$, then

$$P_n(c, d\beta)\{I + MK_{n+1}(c, c, d\alpha)\} = \gamma_n(d\beta)^{-*}\, \gamma_n(d\alpha)^* P_n(c, d\alpha).$$

Since M and $K_{n+1}(c, c, d\alpha)$ are positive definite, we have

$$I + MK_{n+1}(c, c, d\alpha) = (K_{n+1}^{-1}(c, c, d\alpha) + M)\, K_{n+1}(c, c, d\alpha)$$

is non-singular because $I + MK_{n+1}(c, c, d\alpha)$ is the product of two positive definite matrices. Then

$$P_n(c, d\beta) = \gamma_n(d\beta)^{-*}\, \gamma_n(d\alpha)^*\, P_n(c, d\alpha)\{I + MK_{n+1}(c, c, d\alpha)\}^{-1}.$$

Finally

$$P_n(x, d\beta) = \gamma_n(d\beta)^{-*}\, \gamma_n(d\alpha)^*\, P_n(x, d\alpha) - \gamma_n(d\beta)^{-*}\, \gamma_n(d\alpha)^*$$
$$\times P_n(c, d\alpha)\{I + MK_{n+1}(c, c, d\alpha)\}^{-1}\, MK_{n+1}^*(c, x, d\alpha).$$

This means that

$$P_n(x, d\beta) = \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha) \times [P_n(x, d\alpha)$$
$$- P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MK_{n+1}^*(c, x, d\alpha)]. \quad \blacksquare$$

THEOREM 3.1. *Let $\gamma_n(d\beta)$ and $\gamma_n(d\alpha)$ be the leading coefficients of the nth orthonormal matrix polynomials associated to the matrix measures $d\beta$ and $d\alpha$ related by $d\beta(u) = d\alpha(u) + M\delta(u - c)$, where $c \in \mathbb{R} \backslash \hat{\Gamma}$ and M is a positive definite matrix. Assume that*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \qquad \text{and} \qquad \lim_{n \to \infty} E_n(d\alpha) = E,$$

where $D_n(d\alpha)$ and $E_n(d\alpha)$ are the matrix coefficients in the recurrence relation (3), *and D is a non-singular matrix. Then*

$$\lim_{n \to \infty} [\gamma_n(d\beta)\, \gamma_n(d\alpha)^{-1}]^*\, [\gamma_n(d\beta)\, \gamma_n(d\alpha)^{-1}]$$
$$= I + \left(\int \frac{dW_{D, E}(t)}{c - t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{D, E}(t)}{x - t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{D, E}(t)}{c - t} \right). \qquad (15)$$

To prove this, we start with the following lemma.

LEMMA 3.2. *Let $(P_n(x, d\alpha))_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$. Let $D_n(d\alpha)$ and $E_n(d\alpha)$ be the matrix coefficients which appear in the recurrence relation* (3), *satisfying*

$$\lim_{n \to \infty} D_n(d\alpha) = D \qquad \text{and} \qquad \lim_{n \to \infty} E_n(d\alpha) = E,$$

where D is non-singular, then

$$\lim_{n \to \infty} P_n^{-1}(x, d\alpha) = 0, \tag{16}$$

for $x \in \mathbb{R} \backslash \hat{\Gamma}$.

Proof. Notice that from (4), and using (5), we have

$$\begin{aligned}
&P_n^{-1}(x, d\alpha)\, D_n^{-1}(d\alpha)\, P_{n-1}^{-*}(x, d\alpha)\\
&\quad = P_n^{-1}(x, d\alpha)(Q_n(x, d\alpha)\, P_{n-1}^{*}(x, d\alpha)\\
&\qquad - P_n(x, d\alpha)\, Q_{n-1}^{*}(x, d\alpha))\, P_{n-1}^{-*}(x, d\alpha)\\
&\quad = P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha) - (P_{n-1}^{-1}(x, d\alpha)\, Q_{n-1}(x, d\alpha))^*.
\end{aligned}$$

But from (4), we have

$$\begin{aligned}
&P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha)\\
&\quad = \int P_n^{-1}(x, d\alpha)\, \frac{P_n(x, d\alpha) - P_n(t, d\alpha)}{x - t}\, d\alpha(t)\\
&\quad = \int P_n^{-1}(x, d\alpha)\, \frac{P_n(x, d\alpha) - P_n(t, d\alpha)}{x - t}\, d\alpha(t)\\
&\qquad \times (P_n^{*}(x, d\alpha) - P_n^{*}(t, d\alpha))\, P_n^{-*}(x, d\alpha)\\
&\quad = \int d\alpha(t)\, \frac{P_n^{*}(x, d\alpha) - P_n^{*}(t, d\alpha)}{x - t}\, P_n^{-*}(x, d\alpha)\\
&\quad = (P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha))^*.
\end{aligned}$$

Since $(P_n^{-1}(x, d\alpha)\, Q_n(x, d\alpha))_n$ is a convergent sequence for $x \in \mathbb{C} \backslash \Gamma$ (see [3]), then we have

$$\lim_{n \to \infty} P_n^{-1}(x, d\alpha)\, D_n^{-1}(d\alpha)\, P_{n-1}^{-*}(x, d\alpha) = 0. \tag{17}$$

Let $(\varphi(n))_n$ be an increasing sequence of integer numbers such that the limit

$$L(x, d\alpha) = \lim_{n \to \infty} P_{\varphi(n)}^{-1}(x, d\alpha)$$

exists or is ∞, ($L(x, d\alpha) = \infty$ means that at least one of its entries is ∞). Then

$$\begin{aligned} &P^{-1}_{\varphi(n)}(x, d\alpha)\, D^{-1}_{\varphi(n)}(d\alpha)\, P^{-*}_{\varphi(n)-1}(x, d\alpha) \\ &\quad = P^{-1}_{\varphi(n)}(x, d\alpha)\, D^{-1}_{\varphi(n)}(d\alpha)\, P^{-*}_{\varphi(n)-1}(x, d\alpha)\, P^{*}_{\varphi(n)}(x, d\alpha)\, P^{-*}_{\varphi(n)}(x, d\alpha) \\ &\quad = P^{-1}_{\varphi(n)}(x, d\alpha)\, D^{-1}_{\varphi(n)}(d\alpha).(P_{\varphi(n)}(x, d\alpha)\, P^{-1}_{\varphi(n)-1}(x, d\alpha))^*\, P^{-*}_{\varphi(n)}(x, d\alpha). \end{aligned} \tag{18}$$

But from (9), we get

$$\lim_{n\to\infty} (P_{\varphi(n)}(x, d\alpha)\, P^{-1}_{\varphi(n)-1}(x, d\alpha))^* = \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-*} D^{-*}.$$

Using (17) and (18), we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} P^{-1}_{\varphi(n)}(x, d\alpha)\, D^{-1}_{\varphi(n)}(d\alpha)\, P^{-*}_{\varphi(n)-1}(x, d\alpha) \\ &= L(x, d\alpha)\, D^{-1} \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-*} D^{-*} L(x, d\alpha)^* \\ &= (L(x, d\alpha)\, D^{-1}) \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-*} (L(x, d\alpha)\, D^{-1})^* \\ &= \left(L(x, d\alpha)\, D^{-1} \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-1}\right) \\ &\quad \times \int \frac{dW_{D,E}(t)}{x-t} \times \left(L(x, d\alpha)\, D^{-1} \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-1}\right)^*. \end{aligned}$$

Since $x \in \mathbb{R}\backslash\hat{\Gamma}$ and $supp(dW_{D,E}) = supp(d\alpha) \subset \hat{\Gamma}$ we deduce from Proposition 2.2, that the Markov matrix function $\int (dW_{D,E}(t))/(x-t)$ is positive or negative definite. Hence $L(x, d\alpha)\, D^{-1}(\int (dW_{D,E}(t))/(x-t))^{-1} = 0$, and thus

$$L(x, d\alpha) = 0.$$

Hence $P^{-1}_n(x, d\alpha)$ has no subsequence that converges to a number (or ∞) other than 0. Hence (16) holds.

Proof (of Theorem 3.1). We proceed in several steps

Step 3.1.1. Writing $\Phi_n(c) = [\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}]^*\,[\gamma_n(d\beta)\,\gamma_n(d\alpha)^{-1}]$, we have

$$\begin{aligned} \Phi_n(c) = I - [&P_n^{-*}(c, d\alpha)\, M^{-1} P_n^{-1}(c, d\alpha) \\ &+ P_n^{-*}(c, d\alpha)\, K_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha)]^{-1}, \\ \sqrt{N} > \|\gamma_n&(d\beta)\,\gamma_n(d\alpha)^{-1}\|_E \end{aligned} \tag{19}$$

where $\|.\|_E$ is the Frobenius norm.

Proof. If we use Lemma 3.1, we have

$$\begin{aligned} &\int P_n(x, d\beta)\, d\alpha(x)\, P_n^*(x, d\alpha) \\ &\quad = \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha) \Big\{ I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} M \\ &\qquad \times \int K_{n+1}^*(c, x, d\alpha)\, d\alpha(x)\, P_n^*(x, d\alpha) \Big\}. \end{aligned}$$

This means that

$$\begin{aligned} \gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha) = \gamma_n^{-*}&(d\beta)\, \gamma_n^*(d\alpha) \\ &\{I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MP_n^*(c, d\alpha)\}, \end{aligned}$$

and so

$$\begin{aligned} &[\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^*\, [\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)] \\ &\qquad = I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MP_n^*(c, d\alpha). \end{aligned}$$

Hence,

$$\begin{aligned} &I - \Phi_n(c) \\ &\quad = P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MP_n^*(c, d\alpha) \\ &\quad = [P_n^{-1}(c, d\alpha)]^{-1}\, (I + MK_{n+1}(c, c, d\alpha))^{-1}\, [P_n^{-*}(c, d\alpha)\, M^{-1}]^{-1} \\ &\quad = [P_n^{-1}(c, d\alpha) + MK_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha)]^{-1}\, [P_n^{-*}(c, d\alpha)\, M^{-1}]^{-1} \\ &\quad = [P_n^{-*}(c, d\alpha)\, M^{-1} P_n^{-1}(c, d\alpha) + P_n^{-*}(c, d\alpha)\, K_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha)]^{-1}. \end{aligned}$$

Now M and $K_n(c, c, d\alpha)$ are positive definite, hence $I - \Phi_n(c)$ is also positive definite and thus $\Phi_n(c) < I$.

Let $\|.\|_E$ and $\|.\|_s$ be respectively the Frobenius and the spectral norm defined by

$$\|A\|_E^2 = \sum_{i,j=1}^{N} |a_{i,j}|^2 \qquad \text{and} \qquad \|A\|_s^2 = \mu_{(A^*A)},$$

where μ_L is the spectral radius of L ($\mu_L = \max_{1 \leqslant j \leqslant N} |\lambda_j|$; λ_j is an eigenvalue of L). These matrix norms satisfy

- $\|A\|_E^2 = tr(A^*A)$.
- If A is hermitian, then $\|A\|_s = \mu_A$.

Finally,

$$\|\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)\|_E^2 = tr(\Phi_n(c)) \leqslant N\mu_{\Phi_n(c)} = N\,\|\Phi_n(c)\|_s < N.$$

Hence (19) holds. ∎

Step 3.1.2.

$$\lim_{n\to\infty} P_n^{-*}(c, d\alpha)\, K_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha)$$
$$= -\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} \left(\int \frac{dW_{D,E}(t)}{c-t}\right)' \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}. \tag{20}$$

Proof. If we put $P_n^{-*}(c, d\alpha)\, K_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha) = \Upsilon_n(c)$, taking into account (7) and

$$P_n(x)\, P_n^{-1}(x) = I \Rightarrow (P_n(x))'\, P_n^{-1}(x) = -P_n(x)(P_n^{-1}(x))',$$

then we have

$$\begin{aligned}
\Upsilon_n(c) = {} & P_n^{-*}(c, d\alpha)[P_n^*(c, d\alpha)\, D_{n+1}(P_{n+1}(c, d\alpha))' \\
& - P_{n+1}^*(c, d\alpha)\, D_{n+1}^*(P_n(c, d\alpha))']\, P_n^{-1}(c, d\alpha) \\
= {} & D_{n+1}(P_{n+1}(c, d\alpha))'\, P_n^{-1}(c, d\alpha) \\
& - P_n^{-*}(c, d\alpha)\, P_{n+1}^*(c, d\alpha)\, D_{n+1}^*(P_n(c, d\alpha))'\, P_n^{-1}(c, d\alpha) \\
= {} & D_{n+1}(P_{n+1}(c, d\alpha))'\, P_n^{-1}(c, d\alpha) \\
& + P_n^{-*}(c, d\alpha)\, P_{n+1}^*(c, d\alpha)\, D_{n+1}^*\, P_n(c, d\alpha)(P_n^{-1}(c, d\alpha))' \\
= {} & D_{n+1}(P_{n+1}(c, d\alpha))'\, P_n^{-1}(c, d\alpha) \\
& + P_n^{-*}(c, d\alpha)\, P_n^*(c, d\alpha)\, D_{n+1}\, P_{n+1}(c, d\alpha)(P_n^{-1}(c, d\alpha))'
\end{aligned}$$

$$
\begin{aligned}
&= D_{n+1}((P_{n+1}(c, d\alpha))' P_n^{-1}(c, d\alpha) + P_{n+1}(c, d\alpha)(P_n^{-1}(c, d\alpha))') \\
&= D_{n+1}(P_{n+1}(c, d\alpha) P_n^{-1}(c, d\alpha))' \\
&= D_{n+1}[(P_n(c, d\alpha) P_{n+1}^{-1}(c, d\alpha))^{-1}]' \\
&= -D_{n+1}(P_n(c, d\alpha) P_{n+1}^{-1}(c, d\alpha))^{-1} \\
&\quad \times (P_n(c, d\alpha) P_{n+1}^{-1}(c, d\alpha))' (P_n(c, d\alpha) P_{n+1}^{-1}(c, d\alpha))^{-1}.
\end{aligned}
$$

Using (9) and (12), we have

$$
\begin{aligned}
&\lim_{n\to\infty} P_n^{-*}(c, d\alpha) K_{n+1}(c, c, d\alpha) P_n^{-1}(c, d\alpha) \\
&\qquad = -\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} \left(\int \frac{dW_{D,E}(t)}{c-t}\right)' \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}.
\end{aligned}
$$

Thus, for the proof of Theorem 3.1, we can use Lemma 3.2 to find $\lim_{n\to\infty} P_n^{-1}(c, d\alpha) = 0$. Finally, from

$$
\| P_n^{-*}(c, d\alpha) M^{-1} P_n^{-1}(c, d\alpha)\|_s \leqslant \| P_n^{-1}(c, d\alpha)\|_s^2 . \|M^{-1}\|_s,
$$

where $\|.\|_s$ is the spectral norm, we get

$$
\lim_{n\to\infty} P_n^{-*}(c, d\alpha) M^{-1} P_n^{-1}(c, d\alpha) = 0. \tag{21}
$$

Thus (15) follows. ∎

Let $(P_n(x, d\alpha) = \gamma_n(d\alpha) x^n + b_n(d\alpha) x^{n-1} + \textit{lower degree terms})_n$ be a sequence of orthonormal matrix polynomials satisfying (13), then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta) = \gamma_n(d\beta) x^n + b_n(d\beta) x^{n-1} + \textit{lower degree terms})_n$ with respect to the matrix measure $d\beta$ for which $[\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]^*$ are lower triangular matrices with positive diagonal elements. In fact, if from a sequence $(P_n(x, d\beta))_n$ the ratio of leading coefficients $[\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]$ are not upper triangular matrices with positive diagonal elements. Then we can find a sequence of unitary matrices $(S_n)_n$ such that $(S_n^* P_n(x, d\beta) = \lambda_n(d\beta) x^n + \textit{lower degree terms})_n$ is a sequence of othonormal matrix polynomials for which $[\lambda_n(d\beta) \gamma_n^{-1}(d\alpha)]^*$ are lower triangular matrices with positive diagonal elements. To see this, we recall that the associated orthonormal matrix polynomials with respect to the matrix measure $d\alpha$ and $d\beta$ have, respectively, the form $(U_n P_n(x, d\alpha))_n$ and $(V_n P_n(x, d\beta))_n$ (U_n and V_n are unitary matrices). Then the leading coefficients are related by

$$
\begin{aligned}
\lambda_n(d\alpha) &= U_n \gamma_n(d\alpha) \\
\lambda_n(d\beta) &= V_n \gamma_n(d\beta).
\end{aligned}
$$

Therefore

$$[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\;V_n^* = U_n^*[\lambda_n(d\beta)\,\lambda_n^{-1}(d\alpha)]^* \tag{22}$$

and

$$\begin{aligned}[\lambda_n(d\beta)\,\lambda_n^{-1}(d\alpha)]^*\,[\lambda_n(d\beta)\,\lambda_n^{-1}(d\alpha)]\\ = U_n[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]\;U_n^*.\end{aligned} \tag{23}$$

Taking $U_n = I$, (13) holds. Now, we consider the sequence of orthonormal matrix polynomials $(S_n^*\,P_n(x, d\beta))_n$ where $(S_n)_n$ are unitary matrices given by the QR factorization of Francis and Kublanovskaja [10] of $[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]$,

$$[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)] = \tilde{S}_n\tilde{R}_n,$$

where $\tilde{S}_n$ is an unitary matrix and $\tilde{R}_n$ is an upper triangular matrix. Then taking $S_n = \tilde{S}_n J_n$ and $R_n = J_n\tilde{R}_n$, where

$$[J_n]_{i,j} = \begin{cases}\dfrac{[\tilde{R}_n]_{i,i}}{|[\tilde{R}_n]_{i,i}|} & \text{if} \quad i=j\\[2ex] 0 & \text{otherwise,}\end{cases}$$

we get that $[\lambda_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*$ are lower triangular matrices with positive diagonal elements, since from (22) we have

$$[\lambda_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^* = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\;S_n = \tilde{R}_n^* J_n.$$

Now, using (19), let n_ν be an increasing sequence of positive integer numbers such that the limit

$$\Lambda(c) = \lim_{\nu\to\infty}\;[\gamma_{n_\nu}(d\beta)\,\gamma_{n_\nu}^{-1}(d\alpha)]^*$$

exists. From (15), we have

$$\Lambda(c)\,\Lambda^*(c) = I + \int\frac{dW_{D,E}(t)}{c-t}\left\{\left(\int\frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1}\int\frac{dW_{D,E}(t)}{c-t}. \tag{24}$$

The matrix valued function

$$I+\int \frac{dW_{D,E}(t)}{c-t}\cdot\left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1}\cdot\int \frac{dW_{D,E}(t)}{c-t}$$

$$=I-\int \frac{dW_{D,E}(t)}{c-t}\cdot\left\{\int \frac{dW_{D,E}(t)}{(c-t)^2}\right\}^{-1}\cdot\int \frac{dW_{D,E}(t)}{c-t} \tag{25}$$

is positive definite when $c\in\mathbb{R}\backslash\hat{\Gamma}$. In fact from (24), it is sufficient to prove that this matrix is non-singular.

We suppose that there is a non zero vector column x such that

$$\left(I+\int \frac{dW_{D,E}(t)}{c-t}\cdot\left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1}\cdot\int \frac{dW_{D,E}(t)}{c-t}\right)x=0.$$

Then

$$\left(\int \frac{dW_{D,E}(t)}{c-t}\cdot\left\{\int \frac{dW_{D,E}(t)}{(c-t)^2}\right\}^{-1}\cdot\int \frac{dW_{D,E}(t)}{c-t}\right)x=x,$$

and since $\int (dW_{D,E}(t))/(c-t)$ and its derivative are non-singular when $c\in\mathbb{R}\backslash\hat{\Gamma}$, we have

$$\int \frac{dW_{D,E}(t)}{(c-t)^2}\cdot\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}x=\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{2}\cdot\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}x. \tag{26}$$

Writing

$$\int \frac{dW_{D,E}(t)}{c-t}=G_{D,E}(c)$$

$$\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}x=y,$$

then (26) becomes

$$G'_{D,E}(c)\,y=-(G_{D,E}(c))^2\,y;\qquad y\neq 0. \tag{27}$$

From (3), we have

$$(tI-E_n(d\alpha))\,P_n(t,d\alpha)=D_{n+1}(d\alpha)\,P_{n+1}(t,d\alpha)+D_n^*(d\alpha)\,P_{n-1}(t,d\alpha)$$

and then

$$\begin{aligned}D_n^*(d\alpha)(P_{n-1}(t, d\alpha)\, P_n^{-1}(t, d\alpha)\, D_n^{-1}(d\alpha))\\ D_n(d\alpha)(P_n(t, d\alpha)\, P_{n+1}^{-1}(t, d\alpha)\, D_{n+1}^{-1}(d\alpha))\\ +(E_n(d\alpha)-tI)(P_n(t, d\alpha)\, P_{n+1}^{-1}(t, d\alpha)\, D_{n+1}^{-1}(d\alpha))+I=0,\end{aligned}$$

for $t \in \mathbb{R}\backslash\Gamma$. Using (13) and (9), we get

$$D^* G_{D, E}(t)\, D G_{D, E}(t)+(E-tI)\, G_{D, E}(t)+I=0. \tag{28}$$

Taking derivatives in (28) at the point c, we get

$$\begin{aligned}D^* G'_{D, E}(c)\, D G_{D, E}(c)+D^* G_{D, E}(c)\, D G'_{D, E}(c)\\ -G_{D, E}(c)+(E-cI)\, G'_{D, E}(c)=0.\end{aligned}$$

Using (27), we have

$$\begin{aligned}D^* G'_{D, E}(c)\, D G_{D, E}(c)\, y-D^* G_{D, E}(c)\, D(G_{D, E}(c))^2 y\\ -G_{D, E}(c)\, y-(E-cI)(G_{D, E}(c))^2 y=0.\end{aligned}$$

This means that

$$\begin{aligned}(D^* G'_{D, E}(c)\, D)\, G_{D, E}(c)\, y-\{D^* G_{D, E}(c)\, D G_{D, E}(c)\\ +(E-cI)\, G_{D, E}(c)+I\}\, G_{D, E}(c)\, y=0,\end{aligned}$$

and therefore

$$(D^* G'_{D, E}(c)\, D)\, x=0.$$

Hence (25) is positive definite when $c \in \mathbb{R}\backslash\hat{\Gamma}$. Using the Cholesky factorization of (25), the limit $\Lambda(c)$ is finite, with positive diagonal elements, and it is unique.

This means that every convergent subsequence of $[\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^*$ converges to $\Lambda(c)$, hence

$$\lim_{n \to \infty} [\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^* = \Lambda(c). \tag{29}$$

Notice that we can also find $(P_n(x, d\beta))_n$ such that $[\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^*$ are upper triangular matrices with positive diagonal elements. In fact, as before, it is sufficient to give the unitary matrices $(S_n)_n$ by using the QL factorization instead of the QR factorization, and since the matrix given in the right hand side of (25) is positive definite, there is a unique upper triangular matrix $\tilde{\Lambda}(c)$, with positive diagonal elements satisfying

$$\tilde{\Lambda}(c).\tilde{\Lambda}(c)^* = I + \int \frac{dW_{D,E}(t)}{c-t} . \left\{ \left(\int \frac{dW_{D,E}(t)}{c-t} \right)' \right\}^{-1} . \int \frac{dW_{D,E}(t)}{c-t}$$

such that

$$\lim_{n \to \infty} [\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^* = \tilde{\Lambda}(c).$$

Now we consider the matrix measure

$$d\beta(u) = d\alpha(u) + \sum_{k=1}^{N} M_k \delta(u - c_k), \tag{30}$$

where M_k are positive definite matrices and $c_k \in \mathbb{R} \backslash \hat{\Gamma}$ as well as the family of the matrix measures $d\beta_n$, $n = 0, ..., N$ defined by

$$\begin{aligned} d\beta_n(u) &= d\alpha(u) + \sum_{k=1}^{n} M_k \delta(u - c_k), \qquad n = 1, ..., N-1, \\ d\beta_0 &= d\alpha, \; d\beta_N = d\beta. \end{aligned} \tag{31}$$

By repeated application of the previous results we obtain asymptotics and estimates for $\gamma_n(d\beta)$.

THEOREM 3.2. *Let $\gamma_n(d\alpha)$ be the leading coefficients of the orthonormal matrix polynomials $(P_n(x, d\alpha) = \gamma_n(d\alpha)\, x^n +$ lower degree terms$)_n$ associated to $d\alpha$. Assume that*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \qquad \textit{and} \qquad \lim_{n \to \infty} E_n(d\alpha) = E,$$

where D is a non-singular matrix. Then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ with respect to $d\beta$ defined by (30) *such that*

$$\lim_{n \to \infty} [\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^* = \Lambda_1(c_1)\, \Lambda_2(c_2) \cdots \Lambda_N(c_N),$$

where

$$\Lambda_k(c_k) = \lim_{n \to \infty} [\gamma_n(d\beta_k)\, \gamma_n^{-1}(d\beta_{k-1})]^*, \qquad \textit{for} \quad k = 1, ..., N. \tag{32}$$

Proof. The family of the matrix measures defined by (31) can be generated in the following way

$$d\beta_{m+1}(u) = d\beta_m(u) + M_{m+1}\delta(u - c_{m+1});\ m = 0, 1, ..., N-1.$$

Since

$$\begin{aligned}\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha) &= \gamma_n(d\beta_N)\,\gamma_n^{-1}(d\beta_0)\\ &= \gamma_n(d\beta_N)\,\gamma_n^{-1}(d\beta_{N-1})\,\gamma_n(d\beta_{N-1})\cdots\gamma_n^{-1}(d\beta_1)\\ &\qquad \gamma_n(d\beta_1)\,\gamma_n^{-1}(d\beta_0),\end{aligned}$$

then from [18, Thm 4.1], if

$$\lim_{n\to\infty} D_n(d\beta_m) = D_m,$$

and

$$\lim_{n\to\infty} E_n(d\beta_m) = E_m;\ D_m \text{ non-singular}$$

for each $m = 0, 1, ..., N-1$, then there is a sequence of orthonormal matrix polynomials with respect to $d\beta_{m+1}$, such that its associated matrix recurrence coefficients satisfy

$$\lim_{n\to\infty} D_n(d\beta_{m+1}) = \Lambda^*_{m+1}(c_{m+1})\, D_m \Lambda^{-*}_{m-1}(c_{m+1})$$

and

$$\begin{aligned}\lim_{n\to\infty} E_n(d\beta_{m+1}) = \Lambda^*_{m+1}(c_{m+1})\\ \Big\{ & E_m + D_m[\Lambda^{-*}_{m-1}(c_{m+1}).\Lambda^{-1}_{m+1}(c_{m+1}) - I_N]\, D^*_m\\ &\times \int \frac{dW_{D_m, E_m}(t)}{c_{m+1}-t}\\ &- [\Lambda^{-*}_{m-1}(c_{m+1}).\Lambda^{-1}_{m+1}(c_{m+1}) - I_N]\, D^*_m\\ &\times \int \frac{dW_{D_m, E_m}(t)}{c_{m+1}-t}\, D_m \Big\}\, \Lambda^{-*}_{m-1}(c_{m+1}).\end{aligned}$$

Hence by repeated application of (29) and taking into account (32) we get

$$\lim_{n\to\infty} [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^* = \Lambda_1(c_1)\,\Lambda_2(c_2)\cdots\Lambda_N(c_N). \quad \blacksquare$$

EXAMPLE 1. We take

$$D=\begin{pmatrix}4 & 0\\ 0 & \frac{1}{9}\end{pmatrix}, \qquad E=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

A straightforward computation yields

$$\frac{1}{2}D^{-1}(cI-E)\,D^{-1}=\begin{pmatrix}\dfrac{-1+c}{32} & 0\\ 0 & \dfrac{81(-1+c)}{2}\end{pmatrix}$$

and

$$D^{-1/2}(E-cI)\,D^{-1}(E-cI)\,D^{-1/2}-4I=\begin{pmatrix}\dfrac{-63-2c+c^2}{16} & 0\\ 0 & 77-162c+81c^2\end{pmatrix}.$$

From the equation (10), we get

$$\int\frac{dW_{D,E}(t)}{c-t}=\frac{1}{2}\begin{pmatrix}a(c) & 0\\ 0 & b(c)\end{pmatrix}, \tag{33}$$

where

$$a(c)=\tfrac{1}{16}(-1+c-\sqrt{-63-2\,c+c^2})$$
$$b(c)=(-81\,(1-c)-9\,\sqrt{77-162\,c+81\,c^2}),$$

$supp(dW_{D,E})=[-7,9]\bigcup[\frac{7}{9},\frac{11}{9}]=[-7,9]$, and the square roots are chosen such that $\int(dW_{D,E}(t))/(z-t)$ is analytic in $\mathbb{C}\backslash[-7,9]$.

Taking derivatives on the right hand-side of the equation (33) and computing its inverse, we get

$$\left(\frac{d}{dz}\int\frac{dW_{D,E}(t)}{z-t}\right)^{-1}(c)=\begin{pmatrix}x_1(c) & 0\\ 0 & x_2(c)\end{pmatrix},$$

where

$$x_1(c)=-\frac{(-9+c)(7+c)+(-1+c)\,\sqrt{(-9+c)\,(7+c)}}{2}$$

$$x_2(c)=-\frac{77}{162}+c-\frac{c^2}{2}-\frac{(-1+c)\,\sqrt{(-11+9\,c)\,(-7+9\,c)}}{18}.$$

Computing the terms given in (24), we get

$$\Lambda(c)\,\Lambda^*(c) = \begin{pmatrix} y_1(c) & 0 \\ 0 & y_2(c) \end{pmatrix},$$

where

$$y_1(c) = \tfrac{1}{32}(-31 - 2\,c + c^2 - (-1 + c)\,\sqrt{(-9 + c)\,(7 + c)})$$
$$y_2(c) = \tfrac{1}{2}(79 - 162\,c + 81\,c^2 - 9\,(-1 + c)\,\sqrt{(-11 + 9\,c)\,(-7 + 9\,c)}).$$

Finally, using the Cholesky decomposition, we obtain

$$\Lambda(c) = \frac{1}{4\sqrt{2}} \begin{pmatrix} z_1(c) & 0 \\ 0 & z_2(c) \end{pmatrix}, \tag{34}$$

where

$$z_1(c) = \sqrt{-31 - 2\,c + c^2 - (-1 + c)\,\sqrt{(-9 + c)\,(7 + c)}}$$
$$z_2(c) = 4\sqrt{79 - 162\,c + 81\,c^2 - 9\,(-1 + c)\,\sqrt{(-11 + 9\,c)\,(-7 + 9\,c)}}.$$

4. RELATIVE ASYMPTOTICS FOR ORTHONORMAL MATRIX POLYNOMIALS

We will now give some asymptotic results for ratios and products of orthonormal matrix polynomials with respect to the matrix measures $d\alpha$ and $d\beta$ related by $d\beta(u) = d\alpha(u) + M\delta(u - c)$, with $c \in \mathbb{R}\backslash\hat{\Gamma}$ and M a positive definite matrix.

First we will obtain the relative asymptotics for the ratio of some orthonormal matrix polynomials $(P_n(x, d\beta))_n$ and $(P_n(x, d\alpha))_n$ with respect to the matrix measure $d\beta$ and $d\alpha$ respectively.

THEOREM 4.1. *Let $(P_n(x, d\alpha))_n$ be orthonormal matrix polynomials with respect to $d\alpha$. Let $(D_n(d\alpha))_n$ be positive definite and $(E_n(d\alpha))_n$ hermitian matrices which appear in* (3), *satisfying*

$$\lim_{n\to\infty} D_n(d\alpha) = D, \ \lim_{n\to\infty} E_n(d\alpha) = E; \ D \text{ is non-singular.}$$

Then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ *with respect to* $d\beta$ *such that for* $x \in \mathbb{R} \setminus \{\hat{\Gamma} \cup \{c\}\}$

$$\lim_{n\to\infty} P_n(x, d\beta)\, P_n^{-1}(x, d\alpha) = \Lambda(c)^{-1} + \frac{1}{c-x}\,\{\Lambda(c)^* - \Lambda(c)^{-1}\} \times \left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-*} - \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-1}\right\}, \tag{35}$$

where $\Lambda(c)$ *is given in* (29).

Proof. If we multiply in (14) by $P_n^{-1}(x, d\alpha)$, we have

$$\begin{aligned} &P_n(x, d\beta)\, P_n^{-1}(x, d\alpha) \\ &\quad= \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha)[I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} \\ &\qquad \times MK_{n+1}^*(c, x, d\alpha)\, P_n(x, d\alpha)^{-1}]. \end{aligned}$$

But,

$$\begin{aligned} &(c-x)\, K_{n+1}^*(c, x, d\alpha) \\ &\quad= P_{n+1}^*(c, d\alpha)\, D_{n+1}^*\, P_n(x, d\alpha) - P_n^*(c, d\alpha)\, D_{n+1} P_{n+1}(x, d\alpha), \end{aligned}$$

so

$$\begin{aligned} MK_{n+1}^*(c, x, d\alpha)\, P_n^{-1}(x, d\alpha) = \frac{1}{c-x}\, M[&P_{n+1}^*(c, d\alpha)\, D_{n+1}^* \\ &- P_n^*(c, d\alpha)\, D_{n+1} P_{n+1}(x, d\alpha)\, P_n^{-1}(x, d\alpha)], \end{aligned}$$

and hence

$$\begin{aligned} &P_n(x, d\beta)\, P_n^{-1}(x, d\alpha) \\ &\quad= \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha) \left\{ I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} \frac{1}{c-x} M \right. \\ &\qquad \left. \times [P_{n+1}^*(c, d\alpha)\, D_{n+1}^* - P_n^*(c, d\alpha)\, D_{n+1} P_{n+1}(x, d\alpha)\, P_n^{-1}(x, d\alpha)] \right\} \\ &\quad= \gamma_n^{-*}(d\beta)\, \gamma_n^*(d\alpha) \left\{ I - \frac{1}{c-x}\, P_n(c, d\alpha) \right. \\ &\qquad \times (I + MK_{n+1}(c, c, d\alpha))^{-1}\, MP_n^*(c, d\alpha) \\ &\qquad \left. \times [P_n^{-*}(c, d\alpha)\, P_{n+1}^*(c, d\alpha)\, D_{n+1}^* - D_{n+1} P_{n+1}(x, d\alpha)\, P_n^{-1}(x, d\alpha)] \right\}. \end{aligned}$$

Since

$$
\begin{aligned}
&[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]\\
&\qquad = I - P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\,MP_n^*(c, d\alpha)
\end{aligned}
$$

we have

$$
\begin{aligned}
&P_n(x, d\beta)\,P_n^{-1}(x, d\alpha)\\
&\quad = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\left\{I - \frac{1}{c-x}\,I - [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)])\right.\\
&\qquad \left.\times [P_n^{-*}(c, d\alpha)\,P_{n+1}^*(c, d\alpha)\,D_{n+1}^* - D_{n+1}P_{n+1}(x, d\alpha)\,P_n^{-1}(x, d\alpha)]\right\}.
\end{aligned}
\tag{36}
$$

Writing

$$
\begin{aligned}
\Xi_n(x, d\alpha) = {}&[D_{n+1}P_{n+1}(c, d\alpha)\,P_n^{-1}(c, d\alpha)]^*\\
&- [D_{n+1}P_{n+1}(x, d\alpha)\,P_n^{-1}(x, d\alpha)],
\end{aligned}
$$

(36) becomes

$$
\begin{aligned}
&P_n(x, d\beta)\,P_n^{-1}(x, d\alpha)\\
&\quad = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\\
&\qquad \times \left\{I - \frac{1}{c-x}\left(I - [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^*\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]\right)\Xi_n(x, d\alpha)\right\}\\
&\quad = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*} - \frac{1}{c-x}\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\;\Xi_n(x, d\alpha)\\
&\qquad + \frac{1}{c-x}\,[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\,\Xi_n(x, d\alpha)\\
&\quad = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\\
&\qquad + \frac{1}{c-x}\,\{[\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)] - [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\}\;\Xi_n(x, d\alpha).
\end{aligned}
$$

From (9) we have

$$
\lim_{n\to\infty}\,\Xi_n(x, d\alpha) = \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-*} - \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-1}.
$$

By Theorem 3.2, there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ such that

$$\lim_{n \to \infty} [\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^* = \Lambda(c),$$

hence

$$\lim_{n \to \infty} P_n(x, d\beta)\, P_n^{-1}(x, d\alpha) = \Lambda(c)^{-1} + \frac{1}{c-x} \{\Lambda(c)^* - \Lambda(c)^{-1}\}$$

$$\times \left\{ \left(\int \frac{dW_{D,E}(t)}{c-t} \right)^{-*} - \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-1} \right\}. \quad ▮$$

Now we will generalize Theorem 4.1, assuming that $d\beta$ and $d\alpha$ are related by (30).

THEOREM 4.2. *Let $(P_n(x, d\alpha))_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$. Assume that the matrix coefficients in* (3) *satisfy*

$$\lim_{n \to \infty} D_n(d\alpha) = D, \quad \lim_{n \to \infty} E_n(d\alpha) = E; \; D \textit{ is non-singular.}$$

Then there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ with respect to $d\beta$ such that for $x \in \mathbb{R} \setminus \{\hat{\Gamma} \cup \{c_k; k = 1, ..., N\}\}$

$$\lim_{n \to \infty} P_n(x, d\beta)\, P_n^{-1}(x, d\alpha)$$

$$= \overset{\curvearrowright}{\prod_{k=N}^{1}} \left\{ \Lambda_k(c_k)^{-1} + \frac{1}{c_k - x} [\Lambda_k(c_k)^* - \Lambda_k(c_k)^{-1}] \right.$$

$$\left. \times \left[\left(\int \frac{dW_{D,E}(t)}{c_k - t} \right)^{-*} - \left(\int \frac{dW_{D,E}(t)}{x-t} \right)^{-1} \right] \right\},$$

where $\overset{\curvearrowright}{\prod}_{k=r}^{r+s} T_k = T_r T_{r+1} \cdots T_{r+s}$ and $\Lambda_k(c_k)$ is given by (32).

Proof. This follows immediately from Theorem 4.1 using the proof given in Theorem 3.2. ▮

Next we will obtain an asymptotic formula for the product of the orthonormal matrix polynomials $P_n(c, d\beta)$ and $P_n(c, d\alpha)$ at the mass point c, where $d\beta(u) = d\alpha(u) + M\delta(u - c)$, with M a positive definite matrix and $c \in \mathbb{R} \setminus \hat{\Gamma}$.

THEOREM 4.3. *Let $(P_n(x, d\alpha))_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $d\alpha$. Assume that the matrix coefficients in (3) satisfy*

$$\lim_{n\to\infty} D_n(d\alpha) = D, \lim_{n\to\infty} E_n(d\alpha) = E; \ D \text{ is non-singular.}$$

Then there exits a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ with respect to the matrix measure $d\beta$, such that

$$\lim_{n\to\infty} P_n(c, d\beta)\, MP_n^*(c, d\alpha)$$
$$= -\Lambda(c)^{-1} \left\{\int \frac{dW_{D,E}(t)}{c-t}\right\}\left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1} \left\{\int \frac{dW_{D,E}(t)}{c-t}\right\}. \tag{37}$$

Proof. Multiplying $MP_n^*(c, d\alpha)$ to the right on the both hand-sides of (14), we have

$$P_n(c, d\beta)\, MP_n^*(c, d\alpha)$$
$$= [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\{P_n(c, d\alpha)\, MP_n^*(c, d\alpha)$$
$$- P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MK_{n+1}(c, c, d\alpha)\, MP_n^*(c, d\alpha)\}.$$

But

$$P_n(c, d\alpha)\, MP_n^*(c, d\alpha)$$
$$= P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MP_n^*(c, d\alpha)$$
$$+ P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1} MK_{n+1}(c, c, d\alpha)\, MP_n^*(c, d\alpha),$$

so

$$P_n(c, d\beta)\, MP_n^*(c, d\alpha) = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}$$
$$\times \{P_n(c, d\alpha)(I + MK_{n+1}(c, c, d\alpha))^{-1}\, MP_n^*(c, d\alpha)\}.$$

This means that

$$P_n(c, d\beta)\, MP_n^*(c, d\alpha) = [\gamma_n(d\beta)\,\gamma_n^{-1}(d\alpha)]^{-*}\,\{P_n^{-*}(c, d\alpha)\, M^{-1}P_n^{-1}(c, d\alpha)$$
$$+ P_n^{-*}(c, d\alpha)\, K_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha)\}^{-1}.$$

As in the proof of (21) we have

$$\lim_{n\to\infty} P_n^{-*}(c, d\alpha)\, M^{-1}P_n^{-1}(c, d\alpha) = 0,$$

and from (20)

$$\lim_{n\to\infty} P_n^{-*}(c, d\alpha)\, K_{n+1}(c, c, d\alpha)\, P_n^{-1}(c, d\alpha)$$

$$= -\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1} \left(\int \frac{dW_{D,E}(t)}{c-t}\right)' \left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-1}.$$

By Theorem 3.2, there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ such that

$$\lim_{n\to\infty} [\gamma_n(d\beta)\, \gamma_n^{-1}(d\alpha)]^* = \Lambda(c),$$

hence

$$\lim_{n\to\infty} P_n(c, d\beta)\, MP_n^*(c, d\alpha)$$

$$= -\Lambda(c)^{-1} \left\{\int \frac{dW_{D,E}(t)}{c-t}\right\} \left\{\left(\int \frac{dW_{D,E}(t)}{c-t}\right)'\right\}^{-1} \left\{\int \frac{dW_{D,E}(t)}{c-t}\right\}.$$

Example 2. We compute the relative ratio asymptotic of orthogonal matrix polynomials whose coefficients of the recurrence relations converge to

$$D = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using (33) in the Example 1, we have

$$\left(\int \frac{dW_{D,E}(t)}{c-t}\right)^{-*} - \left(\int \frac{dW_{D,E}(t)}{x-t}\right)^{-1} = \begin{pmatrix} \alpha(c, x) & 0 \\ 0 & \beta(c, x) \end{pmatrix},$$

where

$$\alpha(c, x) = \frac{c + \sqrt{(-9+c)\,(7+c)} - x - \sqrt{(-9+x)\,(7+x)}}{2}$$

$$\beta(c, x) = \frac{9\,c + \sqrt{(-11+9\,c)\,(-7+9\,c)} - 9\,x - \sqrt{(-11+9\,x)\,(-7+9\,x)}}{18}.$$

Therefore

$$\lim_{n\to\infty} P_n(x, d\beta)\, P_n^{-1}(x, d\alpha) = \begin{pmatrix} \dfrac{\gamma(c, x)}{\lambda(c, x)} & 0 \\ 0 & \dfrac{\mu(c, x)}{\nu(c, x)} \end{pmatrix},$$

where

$$\gamma(c, x) = 64 - \frac{\begin{pmatrix}(63 + 2\,c - c^2 + (-1 + c)\sqrt{(-9 + c)(7 + c)}) \\ \times (c + \sqrt{(-9 + c)(7 + c)} - x - \sqrt{(-9 + x)\,(7 + x)})\end{pmatrix}}{c - x}$$

$$\lambda(c, x) = \frac{16}{\sqrt{2}}\sqrt{-31 - 2c + c^2 - (-1 + c)\sqrt{(-9 + c)(7 + c)}}$$

$$\begin{aligned}\mu(c, x) = {} & -9(-1 + c)(77 + 81c(-1 + x) - 81x) \\ & + \sqrt{(-11 + 9c)(-7 + 9c)}(77 + 81c(-1 + x) - 81x) \\ & - (-11 + 9c)(-7 + 9c)\sqrt{(-11 + 9x)(-7 + 9x)} + \\ & + 9(-1 + c)\sqrt{(-11 + 9c)(-7 + 9c)}\sqrt{(-11 + 9x)(-7 + 9x)}\end{aligned}$$

$$\begin{aligned}\nu(c, x) = {} & \frac{36}{\sqrt{2}}(c - x) \\ & \times \sqrt{79 - 162c + 81c^2 - 9(-1 + c)\sqrt{(-11 + 9c)(-7 + 9c)}}.\end{aligned}$$

EXAMPLE 3. We compute the product of orthogonal matrix polynomials evaluated at the mass point when the coefficients in the recurrence relation converge to

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By computation of the terms given in (10), we have

$$\int \frac{dW_{D,E}(t)}{c - t} = \frac{1}{4}\begin{pmatrix} a(c) & b(c) \\ b(c) & a(c) \end{pmatrix},$$

where

$$\begin{aligned} a(c) &= -2 + 2c - \sqrt{(-4 + c)\,c} - \sqrt{-4 + c^2} \\ b(c) &= -2 - \sqrt{(-4 + c)\,c} + \sqrt{-4 + c^2}, \end{aligned}$$

$supp(dW_{D,E}) = [0, 4] \bigcup [-2, 2] = [-2, 4]$, and the square roots are chosen such that $\int (dW_{D,E}(t))/(z - t)$ is analytic in $z \in \mathbb{C} \backslash [-2, 4]$.

As in the example 1, we obtain

$$A(c) = \begin{pmatrix} u_1(c) & 0 \\ u_2(c) & u_3(c) \end{pmatrix},$$

where

$$u_1(c)=\frac{1}{2}\sqrt{2(-2+c)\,c-(-2+c)\sqrt{(-4+c)\,c}-c\sqrt{-4+c^2}}$$

$$u_2(c)=\frac{4-4c-(-2+c)\sqrt{(-4+c)\,c}+c\sqrt{-4+c^2}}{\sqrt{2(-2+c)\,c-(-2+c)\sqrt{(-4+c)\,c}-c\sqrt{-4+c^2}}}$$

$$u_3(c)=\frac{1}{2}\left(-4-2c+c^2-\frac{(2+c)(2+(-4+c)\,c)}{\sqrt{-4+c^2}}\right.$$
$$\left.+\sqrt{(-4+c)\,c}\sqrt{-4+c^2}-\frac{\sqrt{(-4+c)\,c}(-2+c^2)}{c}\right)^{1/2}.$$

Then, we have

$$\lim_{n\to\infty} P_n(c, d\beta)\, MP_n^*(c, d\alpha)=\begin{pmatrix} w_1(c) & w_2(c) \\ w_3(c) & w_4(c)\end{pmatrix},$$

where

$$w_1(c)=\frac{4-2(-2+c)\,c+(-2+c)\sqrt{(-4+c)\,c}+c\sqrt{-4+c^2}}{\sqrt{2(-2+c)\,c-(-2+c)\sqrt{(-4+c)\,c}-c\sqrt{-4+c^2}}}$$

$$w_2(c)=\frac{4(-1+c)+(-2+c)\sqrt{(-4+c)\,c}-c\sqrt{-4+c^2}}{\sqrt{2(-2+c)\,c-(-2+c)\sqrt{(-4+c)\,c}-c\sqrt{-4+c^2}}}$$

$$4u_1^2(c)\,u_3(c)\,w_3(c)=4(-1+c)+(-2+c)\sqrt{(-4+c)\,c}-c\sqrt{-4+c^2}$$

$$\begin{aligned}4u_1^2(c)\,u_3(c)\,w_4(c)&=4-12c+2c^2+4c^3-c^4\\&\quad-(-2+c)\,c\sqrt{(-4+c)\,c}\sqrt{-4+c^2}\\&\quad+c(1+(-4+c)\,c)\sqrt{-4+c^2}\\&\quad+(-2+c)\sqrt{(-4+c)\,c}\,(-3+c^2).\end{aligned}$$

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